THE DYNAMICS OF OFF-CENTER REFLECTION

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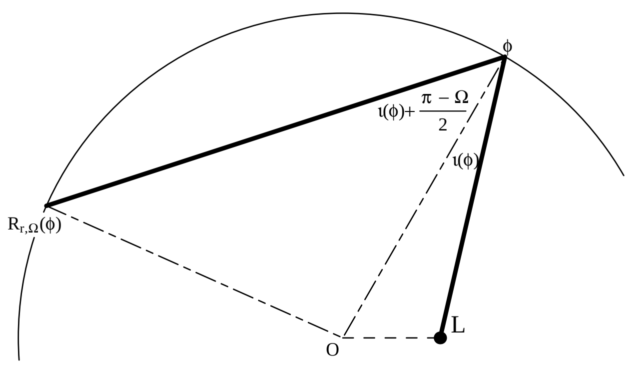
Abstract. Dynamical properties of a two-parameter circle map, called off-center reflection, are studied. Certain symmetry breaking phenomena in the bifurcation process are illustrated and discussed.

Background and Definition

We study the dynamics of a two-parameter family of circle maps $R_{r,\Omega} : \mathbb{S}^1 \to \mathbb{S}^1$ called the off-center reflection. When $\Omega = \pi$, this map is a one-dimensional analog of the general map raised in [Y, problem 21], which is geometrically a reflection in the circle. For other values of Ω, it can be seen as a reflection with a deviation between the reflected and incident angles. Iterations of this map are not the natural sucessive reflections in the circle; nevertheless, this map is interesting for various reasons. The off-center reflection can also be seen as a perturbation (with small $r > 0$) of rotation by Ω. In fact, it has an analytical form which extends the well-known Arnold circle map, [Ar1]. Its dynamics is related to the perturbation properties of Mathieu type differential equation, [Ar2]. Furthermore, when the perturbation parameter r goes to 1, the map goes to another famous circle map, the doubling map.

The off-center reflection is introduced in [AL] by the following geometric description. Fix a point L inside the unit circle $\mathbb{S}^1$. For a point $\phi \in \mathbb{S}^1$, a ray is emitted from L to ϕ. This ray is "reflected" to hit $\mathbb{S}^1$ again at a point, denote $R_{r,\Omega}(\phi)$ in the future. This point is defined to be the image of ϕ under the map. It is quoted "reflected" because the "reflected" angle has a constant deviation $\frac{\pi-\Omega}{2}$ from the incident angle $\iota(\phi)$. To

have the map geometrically well-defined, there is some restriction on Ω. However, we will see later that it is analytically meaningful for other Ω. In fact, it is sufficient to consider $\Omega \in (-\pi, \pi]$. Furthermore, since the action has certain symmetries, it is no loss of generality to assume the point source L at $(r, 0)$ with $0 \leq r < 1$. This is why the off-center reflection is given by the two parameters (r, Ω).



In [AL], particular interest is placed on $R_{r,\pi}$, i.e., when the reflected angle equals the incident angle. There, the link between dynamics and contact geometry of the map is studied. Here, we deal with the dynamical properties of the family with some focus on $\Omega = 0, \pi$. It is because in these two cases, the off-center reflection map repects the symmetry of the circle and symmetric cycles may occur. We are particularly interested in when symmetric cycles of the map break into asymmetric ones. The properties presented in this article may be considered the first steps to understand the dynamics of the map. It is expected that deeper studies may bring forth more understanding to general circle maps.

In §1, we will introduce the basic analytical properties of the map. In §2, the attracting orbits, especially the symmetric ones, of the map are investigated. Then, finally in §3, the bifurcation of the map is looked into. In particular, we give an explanation of how and

when the symmetric orbits go through a period preserving pitch-fork bifurcation. The analysis in [Br] of similar behavior among certain cubic polynomials is borrowed.

1. Analysis

For our purpose, we consider $\mathbb{S}^1 \subset \mathbb{C} \simeq \mathbb{R}^2$ and it is covered by $\mathbb{R}$ under the exponential map $x \mapsto e^{\mathbf{i}x}$. Then the off-center reflection

$$\phi \mapsto R_{r,\Omega}(\phi) : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$$

has a unique continuous lifting, $\tilde{R}_{r,\Omega} : \mathbb{R} \to \mathbb{R}$, which takes 0 to Ω. Since $\tilde{R}_{r,\Omega}(x+2\pi) = \tilde{R}_{r,\Omega}(x) + 2\pi$, we may focus our attention on the interval $(-\pi, \pi]$. Let the incident angle be denoted by $\iota(x)$ for $x \in (-\pi, \pi]$. Since $\iota(x) \to 0$ as $x \to \pm\pi$, it defines a continuous 2π-periodic function on $\mathbb{R}$, which is also denoted as $\iota(x)$. In fact, it can be written in terms of the principal argument (with values between $-\pi$ and π) as

$$\iota(x) = \operatorname{Arg}(\cos x - r + \mathbf{i}\sin x) - x;$$

and $\tilde{R}_{r,\Omega}(x) = x + \Omega - 2\iota(x)$. The first few derivatives of $\tilde{R}$ are listed below.

$$\begin{aligned}
\iota'(x) &= \frac{r(\cos x - r)}{(\cos x - r)^2 + \sin^2 x};\\
\tilde{R}'_{r,\Omega}(x) &= 1 - 2\iota'(x) = \frac{1 - 4r\cos x + 3r^2}{(\cos x - r)^2 + \sin^2 x}\\
\tilde{R}''_{r,\Omega}(x) &= \frac{2r(1-r^2)\sin x}{\left[(\cos x - r)^2 + \sin^2 x\right]^2}\\
\tilde{R}'''_{r,\Omega}(x) &= \frac{2r(1-r^2)\left[(1+r^2)\cos x - 2r(1+\sin^2 x)\right]}{\left[(\cos x - r)^2 + \sin^2 x\right]^3}.
\end{aligned}$$

This incident angle has a series expression given in [AL], $\iota(x) = \sum_{k=1}^{\infty} \frac{r^k}{k} \sin(kx)$. Therefore, the lift of $R_{r,\Omega}$ mapping 0 to Ω is given by

$$\begin{aligned}\tilde{R}_{r,\Omega}(x) &= x + \Omega - 2\iota(x) \\ &= x + \Omega - 2\sum_{k=1}^{\infty} \frac{r^k}{k} \sin(kx).\end{aligned}$$

From this, we see that the Arnold circle map $x \mapsto x+\Omega-\varepsilon \sin x$ can be seen as a truncated version of the off-center reflection. The technique of our analysis in this paper may also be adapted to a similar study on the Arnold circle map.

We will first establish a nice property for the dynamics of a map. The dynamics of the off-center reflection in the range $0 \le r < 1/3$ is relatively simple. The following proposition is useful for studying the range $r > 1/3$.

Proposition 1.1. *For $1/3 < r < 1$ and all Ω, $R_{r,\Omega}$ is a map with negative Schwarzian derivative.*

Proof. The Schwarzian derivative of $\tilde{R}_{r,\Omega}$ is given by

$$\frac{\tilde{R}'''_{r,\Omega}(x)}{\tilde{R}'_{r,\Omega}(x)} - \frac{3}{2}\left(\frac{\tilde{R}''_{r,\Omega}(x)}{\tilde{R}'_{r,\Omega}(x)}\right)^2 = \frac{r(1-r^2)H(r,\cos x)}{(1-4r\cos x+3r^2)^2(1-2r\cos x+r^2)^2}$$

where

$$\begin{aligned}H(r,\cos x) &= (2+28r^2+6r^4)\cos x \\ &\qquad - r\left[13+19r^2-(1-r^2)\cos 2x + 4r\cos 3x\right] \\ &= -(14r+18r^3) + (2+40r^2+6r^4)\cos x \\ &\qquad + 2r(1-r^2)\cos^2 x - 16r^2\cos^3 x\end{aligned}$$

Let $y = \cos x$ and consider $H(r,y)$ for $y \in [-1,1]$. We have

$$\begin{aligned}H(r,-1) &= -2(1+r)^3(1+3r) \\ &= -2-12r-24r^2-20r^3-6r^4 < 0; \\ H(r,1) &= 2(1-r)^3(1-3r) < 0, \qquad \text{for } 1/3 < r < 1.\end{aligned}$$

Moreover, its derivative wrt y satisfies

$$\partial_y H = (2 + 40r^2 + 6r^4) + 4r(1 - r^2)y - 48r^2y^2;$$

and for $1/3 < r < 1$,

$$\partial_y H(r, -1) = 2(1 + r)(1 - r^2)(1 - 3r) < 0,$$

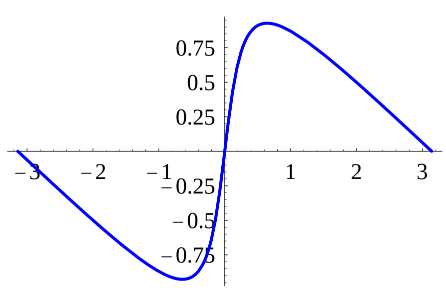
$$\partial_y H(r, 1) = 2(1 - r)(1 - r^2)(1 + 3r) > 0.$$

We see that, inside $(-1, 1)$, the cubic polynomial $y \mapsto H(r, y)$ can only have a single critical point and it is a minimum. Hence $H(r, y) \leq \max\{H(r, -1), H(r, 1)\} < 0$. $\square$

2. Attracting Orbits

It is easy to see from our earlier formula that $\tilde{R}'_{r,\Omega}(x) \geq 0$ for $0 \leq r \leq \frac{1}{3}$, with equality only when $r = \frac{1}{3}$ and $\cos x = 1$. Thus, $R_{r,\Omega}$ is a homeomorphism for $0 \leq r \leq 1/3$, so the dynamics is trivial. On the other hand, $R_{r,\Omega}$ is only a degree 1 map for $r > 1/3$. We would like to explore the dynamics of it in the coming sections.

In the study of periodic orbits of $R_{r,\Omega}$, the information about the function ι is often helpful. Since ι is a 2π-periodic odd function, it is sufficient to know its properties in the interval $[0, \pi]$. To be precise, $\iota(x) > 0$ and is concave down for $x \in (0, \pi)$. Its maximum value of $\pi/2 - a_r$ is attained at a_r where a_r is the angle satisfying $0 \leq a_r \leq \pi/2$ and $\cos a_r = r$. A picture of its graph will be helpful to see its properties.



Furthermore, when r varies from 0 to 1, $\iota(x)$ varies from the constant zero function to a discontinuous linear function. These will be useful in calculations involving periodic orbits of $R_{r,\Omega}$. For example, one may determine the birth of fixed point according to this knowledge of ι.

Proposition 2.1. *For all r and $\Omega \notin [\pi - 2a_r, \pi + 2a_r]$, the map $R_{r,\Omega}$ has no fixed point and a saddle-node bifurcation occurs at $|\Omega - \pi| = 2a_r$, that is, when r is the cosine of the deviation of between incident and reflected angles.*

Proof. To find fixed points of $R_{r,\Omega}$, one tries to solve the equation

$$\tilde{R}_{r,\Omega}(x) = x + \Omega - 2\iota(x) = x \mod 2\pi,$$

which can be reduced to

$$\frac{\Omega}{2} = \iota(x) \mod \pi.$$

Since $\iota(-a_r) \leq \iota(x) \leq \iota(a_r)$, for $\Omega \in (-\pi, \pi]$, the equation has solution if and only if

$$\frac{-\pi}{2} + a_r \leq \frac{\Omega}{2} \leq \frac{\pi}{2} - a_r.$$

The fixed point of $\tilde{R}_{r,\Omega}(x)$ at the boundary parameter values $\Omega = \pm(\pi - 2a_r)$ occurs at $x = a_r$. It is easy to see that for $0 < r < 1$ and any Ω, $\tilde{R}'_{r,\Omega}(a_r) = 1$ and $\tilde{R}''_{r,\Omega}(a_r) \neq 0$, so as (r, Ω) crosses this boundary, a saddle node bifurcation occurs. □

Corollary 2.2. *Let a_r as above and $b_r \in (0, a_r)$ such that $\cos b_r = \dfrac{1+2r^2}{3r}$. The region that $R_{r,\Omega}$ has attracting fixed point is*

$$\{\, (r, \Omega) : 2\iota(b_r) < |\Omega| < 2\iota(a_r) \,\}.$$

In fact, the equation $2\iota(b_r) = |\Omega|$ determines the generic values of r for the happening of period-doubling bifurcation.

Proof. If $x \in (-\pi, \pi]$ corresponds to an attracting fixed point of $R_{r,\Omega}$, we have $\tilde{R}_{r,\Omega}(x) = x$ mod 2π and $\left|\tilde{R}'_{r,\Omega}(x)\right| = |1 - 2\iota'(x)| < 1$. From the expression of $\iota'(x)$, this is equivalent to

$$\cos a_r = r < \cos x < \frac{1 + 2r^2}{3r}.$$

Then, $x \in (-a_r, -b_r) \cup (b_r, a_r)$. Since $\iota(x)$ is increasing on both $(-a_r, -b_r)$ and (b_r, a_r), $\Omega/2$ lies in the intervals defined by the image under ι, namely, $\iota(b_r) < |\Omega/2| < \iota(a_r)$. To see the period doubling bifurcation, it is sufficient to show

$$\left.\frac{\partial(\tilde{R}^2_{r,\Omega})'}{\partial r}\right|_{x=\pm b_r} \neq 0, \qquad \text{along the curves } \Omega = \pm 2\iota(b_r).$$

Here we need some calculations which will also be useful in the future.

Lemma . *We have*

$$\frac{\partial \tilde{R}_{r,\Omega}(x)}{\partial r} = \frac{-2\sin x}{1 - 2r\cos x + r^2},$$
$$\frac{\partial \tilde{R}'_{r,\Omega}(x)}{\partial r} = \frac{4r - 2(1 + r^2)\cos x}{(1 - 2r\cos x + r^2)^2}.$$

Furthermore,

$$\frac{\partial(\tilde{R}^2_{r,\Omega})'(x)}{\partial r} = \frac{\partial \tilde{R}'_{r,\Omega}(x)}{\partial r} \cdot \tilde{R}'_{r,\Omega}(\tilde{R}_{r,\Omega}(x)) + \tilde{R}'_{r,\Omega}(x) \cdot \left.\frac{\partial \tilde{R}'_{r,\Omega}}{\partial r}\right|_{\tilde{R}_{r,\Omega}(x)} \cdot \frac{\partial \tilde{R}_{r,\Omega}(x)}{\partial r}.$$

In particular, if $\tilde{R}_{r,\Omega}(x) = x$ or $\tilde{R}_{r,\Omega}(x) = -x$, one has

$$\frac{\partial(\tilde{R}^2_{r,\Omega})'(x)}{\partial r} = \tilde{R}'_{r,\Omega}(x) \cdot \frac{\partial \tilde{R}'_{r,\Omega}(x)}{\partial r} \cdot \left[1 + \frac{\partial \tilde{R}_{r,\Omega}(x)}{\partial r}\right].$$

Proof of the lemma. The first two results only require simple calculus. The third one is a repeated application of the chain rule. Then, using that both $\tilde{R}'_{r,\Omega}$ and $\dfrac{\partial \tilde{R}'_{r,\Omega}}{\partial r}$ are even functions in x, the last result follows. □

At $\Omega = \pm\iota(b_r)$, we have $\tilde{R}_{r,\Omega}(b_r) = b_r$ and $\tilde{R}'_{r,\Omega}(b_r) = -1$, therefore

$$\left.\frac{\partial(\tilde{R}^2_{r,\Omega})'}{\partial r}\right|_{x=\pm b_r} = -\left.\frac{\partial(\tilde{R}_{r,\Omega})'}{\partial r}\right|_{x=b_r} \cdot \left[1 + \left.\frac{\partial \tilde{R}_{r,\Omega}}{\partial r}\right|_{x=b_r}\right]$$

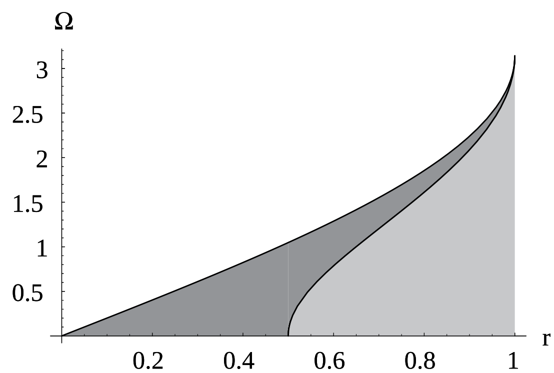
Moreover, for $r > 1/2$,

$$\left.\frac{\partial \tilde{R}'_{r,\Omega}}{\partial r}\right|_{x=b_r} = \frac{4r - 2(1+r^2)\cos b_r}{(1-2r\cos b_r + r^2)^2} = \frac{-6(1-2r^2)}{r(1-r^2)} \neq 0.$$

$$\left.\frac{\partial \tilde{R}_{r,\Omega}}{\partial r}\right|_{x=b_r} = \frac{-2\sin b_r}{1-2r\cos b_r + r^2} = \frac{-2\sqrt{4r^2-1}}{r\sqrt{1-r^2}} \neq -1,$$

except at $r = \sqrt{\frac{-15+\sqrt{241}}{2}} \approx 0.5119$. Thus, generically, period-doubling bifurcation occurs at $\Omega = \pm 2\iota(b_r)$. □

The above information about fixed points is illustrated by the following picture, in which the grey areas are the region in (r, Ω)-plane where fixed points exist. The darker area is where attracting fixed points occur.



We would like to look into maps in the family that respect the symmetry of the circle. Let $\rho : \mathbb{S}^1 \to \mathbb{S}^1$ be the reflection across the real axis. Then $R_{r,\Omega} \circ \rho = \rho \circ R_{r,\Omega}$ if and only if $\Omega = 0, \pi$. This symmetry corresponds to the fact that $\tilde{R}_{r,\Omega}(x) - \Omega$ is an odd function.

For $\Omega = 0, \pi$, if n is the smallest integer that $R^n_{r,\Omega}(\phi) = \rho(\phi) \neq \phi$, one can easily show that ϕ belongs to a periodic orbit of prime period $2n$. We call it a symmetric orbit of period $2n$. In terms of $\tilde{R}_{r,\Omega}$, this corresponds to $\tilde{R}^n_{r,\Omega}(x) = -x \neq x \mod 2\pi$. An orbit is asymmetric of period $2n$ if $R^{2n}_{r,\Omega}(\phi) = \phi$ but $R^n_{r,\Omega}(\phi) \neq \rho(\phi)$. If an asymmetric orbit is formed by ϕ, then another asymmetric one, called the twin orbit, is formed by $\rho(\phi)$. The twin orbit may be itself when $\rho(\phi) = \phi$. For the off-center reflection, a self-twin asymmetric orbit

must be the 2-cycle $\left\{e^{\mathbf{i}0}, e^{\mathbf{i}\pi}\right\}$. There are numerous articles about symmetric periodic orbits in dynamical systems, especially on continuous types. An early one is [D]. Their attention is on the return map of some reversible mechanic system such as the three-body system.

Proposition 2.3. *For any $r > 1/3$ and $\Omega = 0, \pi$, the map $R_{r,\Omega}$ has either no attracting orbit, or one symmetric attracting orbit, or two (counting multiplicity) asymmetric twin attracting orbits.*

Sketch of proof. We mainly use two properties of $R_{r,\Omega}$ to conclude this. First, each $R_{r,\Omega}$ has negative Schwarzian derivative, the technique of [CE] or [Br] is applicable. Therefore, once the map $\tilde{R}_{r,\Omega}$ has an attracting cycle, at least one critical point of it must be attracted to this attracting orbit. Since $R_{r,\Omega}$ has only two critical points, there are at most two attracting orbits. If the attracting orbit is a reflection symmetric one, then this orbit attracts both critical points. If the orbit is an asymmetric one, its twin orbit attracts another critical point. Multiplicity occurs if there is a self-twin orbit. $\square$

Besides the above general information about symmetric and asymmetric cycles, it is interesting to know two specific cases about it. The first is about self-twin asymmetric 2-cycles.

Proposition 2.4. *A single asymmetric 2-cycle for $R_{r,\Omega}$ occurs if and only if $\Omega = \pi$. The orbit is $\left\{ e^{\mathbf{i}0}, e^{\mathbf{i}\pi} \right\}$ which is attracting when $r < 1/\sqrt{5}$.*

Proof. The orbit for a single asymmetric 2-cycle can be obtained by solving the equation

$$\frac{\Omega \pm \pi}{2} = \iota(x) \mod \pi.$$

The only simultaneous solution exists when $\Omega = \pi$ and $\phi = e^{\mathbf{i}0}, e^{\mathbf{i}\pi}$. Then, whether the orbit is attracting can be easily decided by calculating $\tilde{R}'_{r,\pi}(0)\tilde{R}'_{r,\pi}(\pi)$. □

Secondly, one expects that symmetric cycles occur naturally for $\Omega = 0, \pi$. Do such cycles exist even if the map is not "symmetric"? The following result provides a partial evidence for the answer.

Proposition 2.5. *The map $R_{r,\Omega}$ has a symmetric 2-cycle if and only if $\Omega = 0, \pi$. In both cases, the symmetric 2-cycle is unique. The cycle of $R_{r,0}$ is attracting while that of $R_{r,\pi}$ is repelling.*

Proof. Let $\left\{ e^{\mathbf{i}x}, e^{-\mathbf{i}x} \right\}$ be a symmetric 2-cycle, i.e., $R_{r,\Omega}(e^{\pm\mathbf{i}x}) = e^{\mp\mathbf{i}x}$. Equivalently,

$$x + \frac{\Omega}{2} = \iota(x) \mod \pi,$$

and also,

$$x - \frac{\Omega}{2} = \iota(x) \mod \pi.$$

If we subtract the second equation from the first one, we obtained that $\Omega = 0 \mod \pi$ and $\Omega \in (-\pi, \pi]$. Clearly, $\Omega = 0, \pi$.

For $\Omega = 0$, the trivial solutions $x = 0, \pi$ corresponds to fixed points of $R_{r,0}$. Further calculation on ι shows that the equation $x = \iota(x)$ has only a nontrivial solution for $r > 1/2$. In fact, let $c_1 \in (0, \pi/2)$ such that $\cos c_1 = \dfrac{1}{2r}$, then $\pm c_1$ form a symmetric 2-cycle. Since $\tilde{R}'_{r,0}(c_1)\tilde{R}'_{r,0}(-c_1) = \left[\dfrac{1-3r^2}{r^2}\right]^2$, it follows that the cycle is attracting for $\dfrac{1}{2} < r < \dfrac{1}{\sqrt{2}}$. Moreover, since $|\iota(x)| \leq \pi/2$, one can only obtain trivial solution $0, \pi$ from $x = \iota(x) + k\pi$ for $k \neq 0$.

For $\Omega = \pi$, the above equations always has solution for all r. In fact, its solution within $(-\pi, \pi)$ is given by the following. Let $c_2 \in (\pi/2, \pi)$ satisfy $\cos c_2 = \dfrac{1 - \sqrt{1+8r^2}}{4r}$. We

claim that c_2 and $2\pi - c_2$ are solutions to the equation $x - \frac{\pi}{2} = \iota(x)$ and $x + \frac{\pi}{2} = \iota(x)$ respectively. Taking tangent to both sides and using that $\iota(x) = \text{Arg}(\cos x - r + \mathbf{i}\sin x) - \text{Arg}(\cos x + \mathbf{i}\sin x)$, we have

$$\frac{-\cos x}{\sin x} = \frac{r\sin x}{1 - r\cos x}.$$

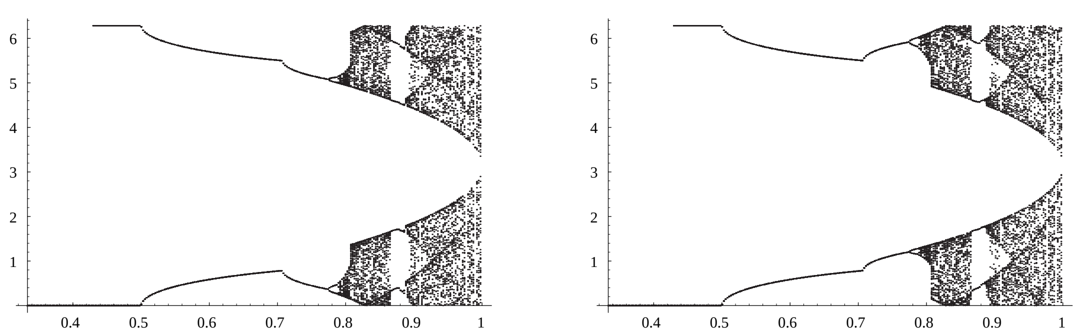
It follows that $2r\cos^2 x - \cos x - r = 0$. Hence $\left\{e^{\mathbf{i}c_2}, e^{-\mathbf{i}c_2}\right\}$ is a symmetric 2-cycle. It can be easily checked that

$$\tilde{R}'_{r,\pi}(c_2) = \tilde{R}'_{r,\pi}(2\pi - c_2) = \frac{2(\sqrt{1+8r^2} + 3r^2)}{1 + \sqrt{1+8r^2} + 2r^2} > 1.$$

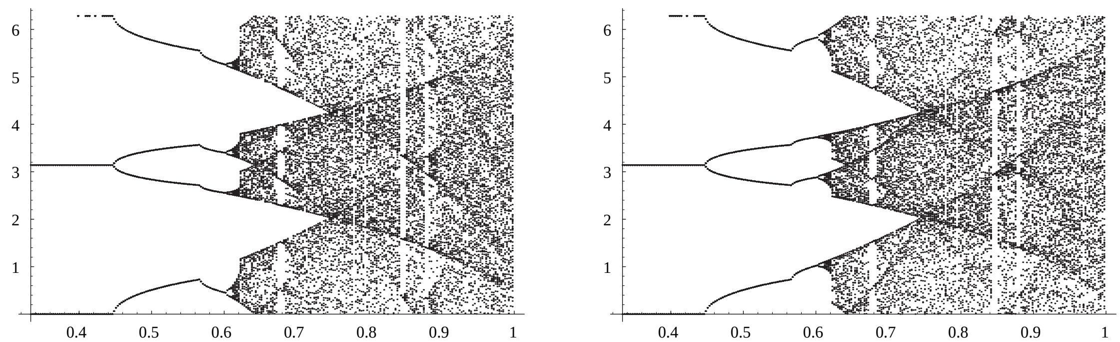
Therefore, the symmetric 2-cycle is repelling. Again, by that $|\iota(x)| \le \pi/2$, we do not have other solutions to $x \pm \frac{\pi}{2} = \iota(x) + k\pi$ for $k \neq 0$. $\square$

3. Bifurcation

In this last section, our aim is to understand bifurcations between symmetric and asymmetric orbits of this family of maps. In particular, we would like to address some of the questions on the dynamics of $R_{r,\pi}$ raised in [AL]. Let us first look at the asymptotic orbit diagrams of the critical points of $R_{r,0}$ and $R_{r,\pi}$, in which the bifurcations are shown.



Asymptotic orbits of critical points of $R_{r,0}$ (low resolution).



Asymptotic orbits of critical points of $R_{r,\pi}$ (low resolution).

In both pictures, there is an obvious complementary nature and there are half-branches of bifurcations. These will be explained analytically in the coming propositions.

Proposition 3.1. *The asymmetric 2-cycle of $R_{r,\pi}$ bifurcates into a symmetric attracting 4-cycle at $r = 1/\sqrt{5}$. For $R_{r,0}$, there is a pitch-fork period preserving bifurcation of the symmetric 2-cycle into asymmetric attracting ones at $r = 1/\sqrt{2}$.*

Proof. For the 2-cycle $\left\{ e^{\mathbf{i}0}, e^{\mathbf{i}\pi} \right\}$ of $R_{r,\pi}$, the derivatives are given by

$$\tilde{R}'_{r,\pi}(0) = \frac{1-3r}{1-r};$$
$$\tilde{R}'_{r,\pi}(\pi) = \frac{1+3r}{1+r}.$$

Thus, the 2-cycle undergoes a period-doubling bifurcation when $-1 = \dfrac{1-9r^2}{1-r^2}$, which is exactly $r = 1/\sqrt{5}$. We claim that the new attracting 4-cycle is reflection symmetric. In fact, the equation for a reflection symmetric 4-cycle of $R_{r,\pi}$ is

$$x + \pi = \iota(x) + \iota \circ \tilde{R}_{r,\pi}(x) \mod \pi.$$

Let $f(x) = x - \iota(x) - \iota(x + \pi - 2\iota(x)) + \pi$. Then we have to solve for $f(x) = 0 \mod \pi$. Clearly, $f(x) - \pi$ is an odd function with $f(-\pi) = 0$, $f(0) = \pi$, and $f(\pi) = 2\pi$. If f is decreasing at $-\pi$, equivalently, it is so at π, then f must have a zero modulo 2π in

neighborhoods of $\pm\pi$. It is easy to compute that

$$\begin{aligned} f'(-\pi) = f'(\pi) &= 1 - \iota'(-\pi) - \iota'(0)\,[1 - 2\iota'(-\pi)] \\ &= \frac{1-5r^2}{1-r^2}. \end{aligned}$$

Thus, $f'(-\pi) \leq 0$ if and only if $r \geq 1/\sqrt{5}$. This shows that for $1/\sqrt{5} < r$, $R_{r,\pi}$ has a symmetric 4-cycle. Moreover, it must be attracting when $r < 1/\sqrt{5} + \varepsilon$ by continuity.

We have shown in the previous section that an attracting symmetric 2-cycle exists for $R_{r,0}$ with $1/2 < r < 1/\sqrt{2}$. It is given by $\tilde{R}_{r,0}(c_1) = -c_1 \mod \pi$. One may further calculate according to the lemma in §2 to obtain that, at $r = 1/\sqrt{2}$ and $x = c_1$,

$$\frac{\partial^2 R^2_{r,0}}{\partial r \partial x} = 2\sqrt{2} - 8 \neq 0.$$

Then, by an argument making use of the Inverse Function Theorem, one concludes that $R_{r,0}$ has a 2-period preserving pitch fork bifurcation at $r = 1/\sqrt{2}$. □

Proposition 3.2. *There is a pitch-fork bifurcation on $R_{r,\pi}$ where a symmetric orbit of period 4 breaks into two asymmetric orbits of period 4.*

Sketch of proof. The key is to consider the zero set of $\tilde{R}^4_{r,\pi}(x) = x$ in the (r, x)-plane. In the above, we have shown that for r slightly larger than $1/\sqrt{5}$, this can be obtained from the solution set of a symmetric 4-cycle, $\tilde{R}^2_{r,\pi}(x) = -x$. We then solve for the specific values of (r_0, x_0) such that $\tilde{R}^2_{r_0,\pi}(x_0) = -x_0$ and $(\tilde{R}^2_{r_0,\pi})'(x_0) = 1$. Our calculation involves the expression of $R_{r,\pi}$ as a Blaschke product given in [AL],

$$R_{r,\pi}(z) = \frac{-z^2(1-rz)}{z-r}.$$

Then, a symmetric 4-cycle can be solved from $z = e^{\mathbf{i}\phi}$ and

$$R^2_{t,\pi}(z) = z^4 \cdot \frac{(1-rz)^2}{(z-r)^2} \cdot \frac{r^2z^3 - rz^2 - z + r}{rz^3 - z^2 - rz + r^2} = \frac{1}{z}.$$

After factoring out the obvious factors $(z+1)(z-1)$, and letting $y = \cos\phi$, we have the polynomial equation

$$-1 - 4r^2 + r^4 + 2r(1 + 7r^3)y + 4r^2(2 - 3r^2)y^2 - 24r^3y^3 + 16r^4y^4 = 0.$$

For $r > 1/\sqrt{5}$, all the four roots of this equation are real. Two of them are always > 1, one lies within $(-1, 0)$ and the other within $(0, 1)$. After taking arccosines, the four solutions form a symmetric 4-cycle. Let $x_0 = x_0(r)$ be a solution, then we solve for r in $(\tilde{R}^2_{r,\pi})'(x_0) = 1$, we obtain a numerical value of r_0 approximately equal to 0.57. We further verify that

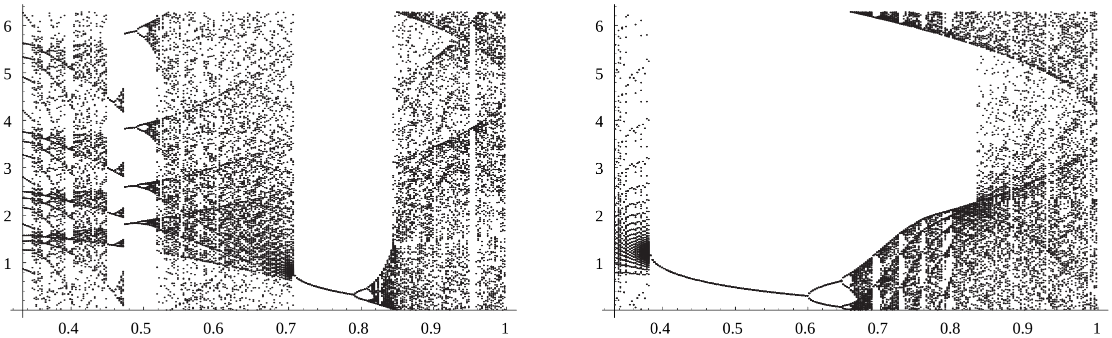
$$\frac{\partial(\tilde{R}^4_{r,\pi})'}{\partial r}(x_0, r_0) \neq 0.$$

Many of the above calculations are lengthy and we indeed make use of computation software to help us. Finally, we apply the Inverse Function Theorem to conclude that there is a 4-period preserving pitch fork bifurcation. □

The local change of the graphs of $R^4_{r,\Omega}$ is shown in the picture.

Graphs of $R^4_{r,\Omega}$ (darker) and $R^8_{r,\Omega}$ (lighter) before and after the pitch-fork bifurcation.

To end this article, we would like to present two pictures, asymptotic orbits for $\Omega = \pi/2, \pi/4$ to illustrate the wide variation of the dynamics of this family with respect to Ω.



Asymptotic orbits of $R_{r,\pi/2}$ and $R_{r,\pi/4}$ (low resolution).

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